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Approximations to x^n and $|x|$—a Survey

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In 1859, almost twenty-five years prior to the publication of Weierstrauss's approximation theorem, the classical theory of the Chebyshev polynomials $\cos(n \cos^{-1} x)$ arose, as we know, from the problem of best uniform approximation to x^n on $[-1, 1]$ by polynomials of degree $<n$. In this connection Chebyshev showed that x^n can be approximated on $[-1, 1]$ by polynomials of degree at most $n-1$ with an error exactly 2^{-n+1}. From his construction, it follows that the best uniformly approximating polynomial of degree at most $(n-1)$ to x^n on $[-1, 1]$ is $P(x) = x^n - 2^{-n+1} T_n(x)$. Here T_n is the Chebyshev polynomial of the nth degree and $P(x)$ is (if $n \geqslant 2$) a polynomial of degree $n-2$, since for n even, all odd coefficients of $T_n(x)$ vanish (but no even) for odd n, the converse holds. As is clear from his proof, x^n cannot be uniformly approximated with error <2 by polynomials of degree at most $n-1$ on any interval whose length is $\geqslant 4$. In 1868, Chebyshev's student Zolotarev extended the above result of Chebyshev as follows. The error obtained in the best uniform approximation of $x^n - \sigma x^{n-1}$ $(0 \leqslant \sigma \leqslant n \tan^2(\pi/2n))$ on $[-1, 1]$ by polynomials of degree at most $n-2$ $(\geqslant 0)$ is $2^{-n+1}(1 + (\sigma/n))^n$. For $\sigma = 0$ and $n \geqslant 2$ we obtain the above stated result of Chebyshev. All real values of σ were handled, using elliptic functions. The polynomials developed by Zolotarev for this purpose played a very significant role in the important investigations of W. A. Markov and N. I. Achieser. Erdös and Szegö [7] proved Zolotarev's result by a different method. In 1976, on my suggestion, Newman [11] has obtained error estimates for the best uniform approximation of x^n on $[-1, 1]$ by rational functions of the form P/Q where P is a polynomial of degree $n-2$ and Q is a polynomial of even degree. In [15], Newman and Reddy have initiated the approximation of x^n on $[0, 1]$ by reciprocals of polynomials of degree

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n and obtained error estimates. Later on, Newman [12], Reddy [18], and Borwein [4] have obtained by different methods better error estimates than those of [15]. In [17] Reddy has shown that an effective approximation to x^n by polynomials of degree k ($k < n$) is possible if and only if $k > \sqrt{n}$ (This result is implicit in Theorem 11 of [17]). In [21] we have observed that if $k/n \to 0$ as $n \to \infty$, then it is not possible to approximate x^n on $[0, 1]$ by reciprocals of polynomials of degree k with an error $\leqslant c^n$ $(0 < c < 1)$. In [21] we have also extended several of the above stated results to the case of several variables, as well as to the case of the union of two disjoint intervals. In [13] Newman and Reddy have studied the problem of approximating x^n on $[0, 1]$ by polynomials and rational functions having only non-negative real coefficients. In this paper we have shown that the least maximal error obtained in approximating x^n on $[0, 1]$ by polynomials of degree $k (1 \leqslant k < n)$ having non-negative real coefficients is equal to the least maximal error obtained by rational functions of the corresponding degree having non-negative real coefficients. In fact, we have established that the best approximating rational function of degree k $(1 \leqslant k < n)$ to x^n on $[0, 1]$ having non-negative real coefficients is nothing but the best approximating polynomial of degree k having non-negative real coefficients. In [20] we have obtained error estimates in approximating x^n on $[-1, 1]$ by rational functions of the form $(P_{n-2}(x)/Q_{2s}(x))$. From these results we get, for the case $s = 0$, the above stated result of Chebyshev. Further, our results improve Newman's, giving sharper estimates. Also we have shown there, that x^n can be approximated uniformly on $[0, 4]$ by rational functions of the form $(P_{n-2}(x)/q_2(x))$ with an error $\leqslant 6/n$.

Now we turn our attention to the approximation of $|x|$ on $[-1, 1]$. As we know, the approximation of $|x|$ on $[-1, 1]$ by polynomials played a very significant role in the early development of approximation theory. In 1908, de la Vallée Poussin raised the question of best approximating $|x|$ on $[-1, 1]$ by polynomials. This problem attracted the attention of several leading mathematicians of that period. Preliminary results were obtained by Lebesgue, de la Vallée–Poussin, Bernstein, and Jackson. In 1911 Bernstein [2] has shown that $|x|$ can be approximated on $[-1, 1]$ by polynomials of degree $2n$ with an error $\leqslant (2n+1)^{-1}$ but not better than $[4(2n-1)(\sqrt{2}+1)]^{-1}$. Finally, in 1913, Bernstein ([5], p. 288) has shown that the least largest error obtained in approximating $|x|$ on $[-1, 1]$ by polynomials of degree $2n$ is asymptotic to $0.282/2n$. In 1964, on a suggestion of Shapiro, Newman [10] has obtained error estimates in approximating $|x|$ on $[-1, 1]$ by rational functions of the form xP/Q, where P and Q are polynomials of degree at most n. In fact, he has shown that $|x|$ can be approximated on $[-1, 1]$ by rational functions of the above form with an error $\leqslant 3e^{-\sqrt{n}}$ for all $n \geqslant 4$, but not better than

$\frac{1}{2}e^{-9\sqrt{n}}$. Later on several Hungarian (Turán and his associates) and Soviet Mathematicians (Gončar and his associates) have obtained generalizations and improvements of the above stated result of Newman. Finally, Bulanov [5] has established that $|x|$ can be approximated on $[-1, 1]$ by rational functions of degree n with an error $e^{-\pi\sqrt{n}}$ but not much better.

In [3] Boehm has shown that $|x|$ can be approximated by reciprocals of polynomials of degree n with an error $\leqslant (1+\pi)n^{-1/3}$. In [9] Lungu, working under the supervision of Gončar, has shown that $|x|$ can be approximated on $[-1, 1]$ by reciprocals of polynomials of degree n with an error $\leqslant n^{-1}$ logn but not better than $(16n)^{-1}$. In [14] Newman and Reddy have shown that $|x|$ can be approximated on $[-1, 1]$ by such reciprocals with an error $\leqslant \pi^2/2n$. In [18] we have initiated the approximation of $\sqrt{1-x}$ on $[0, 1]$.

Chebyshev.

$$\min \|x^n - P_{n-1}(x)\|_{L_\infty[-1,1]} = 2^{-n+1}. \tag{1}$$

Zolotarev. For $0 \leqslant \sigma \leqslant n \tan^2(\pi/2n)$,

$$\min \|x^n - \sigma x^{n-1} - P_{n-2}\|_{L_\infty[-1,1]} = 2^{-n+1}(1+\sigma/n)^n. \tag{2}$$

Achieser ([1, p. 279]). Let $a_0 \neq 0$, $a_1, a_2, \ldots, a_n$ be given real numbers. Then for every $N > n \geqslant 0$,

$$\min_{P_i, q_i} \max_{-1 \leqslant x \leqslant 1} \left| a_0 x^N + \frac{a_1}{2} x^{N-1} + \cdots + \frac{a_n x^{N-n}}{2^n} - \frac{(q_0 x^{N-1} + q_1 x^{N-2} + \cdots + q_{N-1})}{P_0 x^n + \cdots + P_n} \right| = \frac{|\lambda_0|}{2^{N-1}}, \tag{3}$$

where λ_0 is a zero of minimal absolute value of the polynomial

$$\begin{vmatrix} c_n - \lambda & c_{n-1} & \cdots & c_1 & c_0 \\ c_{n-1} & c_{n-2} - \lambda & \cdots & c_0 & 0 \\ \vdots & & & & \\ c_1 & c_0 & & -\lambda & 0 \\ c_0 & 0 & & 0 & -\lambda \end{vmatrix}$$

with $c_m = \sum_{i=0}^{[m/2]} a_{m-2i} \binom{n-m+2i}{i}$, $m = 0, 1, 2, \ldots, n$.

Newman ([11, p. 285]). There is a polynomial $P(x)$ of degree at most $n-1$ and a polynomial $q(x)$ of degree $2s$ such that

$$\left\| x^n - \frac{P(x)}{q(x)} \right\|_{L_\infty[-1,1]} \leqslant 2^{-n+1} \binom{s+n-3}{s}^{-1}. \tag{4}$$

Let $P(x)$ be any polynomial of degree at most $n-1$ and $q(x)$ any polynomial of degree $\leqslant 2s$, then

$$\left\| x^n - \frac{P(x)}{q(x)} \right\|_{L_\infty[-1,1]} \geqslant 2^{-2-n} \binom{s+n+1}{s}^{-1}. \tag{5}$$

Reddy [20]. Given $n \geqslant s+2 \geqslant 4$, there exist polynomials $P(x)$ and $q(x)$ of degrees $n-2$ and $2s$, respectively, for which

$$\left\| x^n - \frac{P(x)}{q(x)} \right\|_{L_\infty[-1,1]} \leqslant \frac{1}{2^{n-1}\{\binom{s+n-3}{s} + \binom{n+s-5}{s-2}\}}. \tag{6}$$

Remark. (6) sharpens (4).

Reddy ([17, Theorems 11 and 18]).

(i) Let $P_{2n-2}(x)$ and $q_{2n-2}(x) = \sum_{i=0}^{2n-2} b_i x^i$, $b_i \geqslant 0$ $(i \geqslant 0)$, be any even polynomials of degrees at most $(2n-2)$. Then

$$\left\| x^{2n} - \frac{P_{2n-2}(x)}{Q_{2n-2}(x)} \right\|_{L_\infty[-1,1]} \geqslant 2^{-2n+1}. \tag{7}$$

(ii) Let $P(x)$ and $q(x)$ be any polynomials of degrees at most $m \geqslant 1$. Then, if $m < 2n$,

$$\left\| x^{2n} - \frac{P(x)}{q(x)} \right\|_{L_\infty[-1,1]} \geqslant 2^{-1} e^{-2\pi(2mn)^{1/2}}. \tag{8}$$

Newman and Reddy ([13, p. 248]). If $P_k(x) = dx^k$, $1 \leqslant k < n$, $d > 0$, and

$$n(1-d) = (n-k)\left(\frac{k}{n}\right)^{k/(n-k)} d^{n/(n-k)}, \tag{9}$$

then $P_k(x)$ is the best uniform approximating polynomial of degree k to x^n on $[0, 1]$. In fact, denoting by ε_k and θ_k respectively, the smallest maximal error in approximating x^n on $[0, 1]$ by polynomials and rational functions having only real non-negative coefficients, we have

$$n\varepsilon_k = (n-k)\left(\frac{k}{n}\right)^{k/(n-k)} (1-\varepsilon_k)^{n/(n-k)}, \tag{10}$$

$$\varepsilon_k = \theta_k = 1-d. \tag{11}$$

Newman and Reddy ([15, p. 452]). (i) For all $n \geqslant 4$

$$\left\| x^n - \left(\sum_{i=0}^{2n-1} \binom{n+i-1}{i} (1-x)^i \right)^{-1} \right\|_{L_\infty[0,1]} \leqslant 16n^2\left(\tfrac{27}{64}\right)^n. \tag{12}$$

(ii) Let $P(x)$ be any polynomial of degree at most m. Then for all $m \geqslant 1$, $n \geqslant 1$,

$$\left\| x^n - \frac{1}{P(x)} \right\|_{L_\infty[0,1]} \geqslant 2^{-n-1}(3+2\sqrt{2})^{-m}. \tag{13}$$

(iii) Let $P(x)$ and $q(x)$ be any real polynomials of degrees at most l $(0 \leqslant l \leqslant n-1)$ and m $(m \geqslant 0)$, respectively. Then (a) for $l = n-1$,

$$\left\| x^n - \frac{P(x)}{q(x)} \right\|_{L_\infty[0,1]} \geqslant \frac{m!\,(2n)!}{(m+2n-1)!\,2^{2n}(m+n)}. \tag{14}$$

(b) For even m,

$$\left\| x^n - \frac{P(x)}{q(x)} \right\|_{L_\infty[0,1]} \geqslant \frac{(m+n-l-1)!(2l+2)!\,2^{-2n-2}}{(m+n+l)!\binom{m+2n-2l}{2n-2l-1}(m-2n)}. \tag{15}$$

Newman ([12, p. 236]). For $0 \leqslant x \leqslant 1$,

$$0 \leqslant \frac{1}{\delta(x)} - x^n \leqslant \frac{2}{k}\left(\frac{2n-2}{2n+k}\right)^{n-1}, \tag{16}$$

where $\delta(x) = \sum_{i=0}^{k} \binom{n+i-1}{i}(1-x)^i$.

Reddy [21].

(i) $$0 \leqslant \left\{ \sum_{i=0}^{2m} \binom{n+i-1}{i}(1-x)^i \right\}^{-1} - x^n \leqslant \binom{n+2m}{2m}\binom{n+m}{m}^{-2}. \tag{17}$$

(ii) Let $P(x)$ be any real polynomial of degree at most m. Then for any constant a $(0 < a < 1)$ we have

$$\left\| x^n - \frac{1}{P(x)} \right\|_{L_\infty[0,1]} \geqslant \frac{a^n}{2T_m\left(\frac{1+a}{1-a}\right)}. \tag{18}$$

Remark. If $m/n \to 0$ as $n \to \infty$, then it is impossible to approximate x^n on $[0, 1]$ by reciprocals of polynomials of degree m with a maximal error $\leqslant C^n$ $(0 < C < 1)$. For set $m = n\delta_n$, $\delta_n \to 0$ as $n \to \infty$ and choose for each n, $a = 1 - \delta_n$. Then

$$T_m\left(\frac{1+a}{1-a}\right) = T_m\left(\frac{2-\delta_n}{\delta_n}\right) < T_m\left(\frac{2}{\delta_n}\right) \leqslant 4^m\,\delta_n^{-m}. \tag{19}$$

Therefore

$$\left\| x^n - \frac{1}{P(x)} \right\|_{L_\infty[0,1]}^{1/n} \geqslant (1-\delta_n)[\tfrac{1}{2} 4^{-m} \delta_n^m]^{1/n} \tag{20}$$

which $\to 1$ as $n \to \infty$.
Hence geometric convergence fails.

Newman and Rivlin ([16, p. 454]).

$$\frac{p_{k,n}}{4e} \leqslant E_n(x^k) \leqslant p_{k,n}. \tag{21}$$

where $0 \leqslant n < k$,

$$p_{k,n} = 2^{-k+1} \sum_{2j > n+k} \binom{k}{j}, \quad E_n(x^k) = \min_{a_j} \left\| x^k - \sum_{j=0}^{n} a_j x^j \right\|_{L_\infty[-1,1]}.$$

Reddy ([17, p. 101]). For any real constant σ,

$$\frac{p_{k,n} + |\sigma| p_{k-1,n}}{\sqrt{2k-2n}} \leqslant E_n(x^k + \sigma x^{k-1}) \leqslant p_{k,n} + |\sigma| p_{k-1,n}. \tag{22}$$

Borwein ([4, p. 241]).
Let $n \geqslant 1$, $m \geqslant 1$. For some real polynomial $P_m(x)$ of degree $\leqslant m$,

$$\left\| x^n - \frac{1}{P_m(x)} \right\|_{L_\infty[0,1]} \leqslant \frac{(4.72)\, n}{\sqrt{2n-1}} \frac{(n+m)!\,(3n-2)!}{(n-1)!\,(3n+m-1)!},$$

while for every such polynomial $q_m(x)$,

$$\left\| x^n - \frac{1}{q_m(x)} \right\|_{L_\infty[0,1]} \geqslant \frac{0.18}{\sqrt{2n+1}} \frac{(n+m)!\,(3n)!}{(n-1)!\,(3n+m+1)!}.$$

Also

$$\lim_{n\to\infty} \left\{ \min_{a_j \text{ real}} \left\| x^n - \left(\sum_{j=0}^{n} a_j x^j \right)^{-1} \right\|_{L_\infty[0,1]} \right\}^{1/n} = \tfrac{27}{64}$$

Reddy [21]. Let $k \geqslant 1$. Then $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_k^{n_k}$ can be approximated in the k-dimensional unit cube by a polynomial P in the variables $x_1, x_2 \cdots, x_k$ with the corresponding degrees at most $n_1, n_2, \ldots, n_k$, respectively, and the total degree of the polynomial not exceeding $n_1+n_2+\cdots+n_k-2$ with a maximal error 2^{-N+k}. In otherwords

$$\max_{-1\leqslant x_j\leqslant 1,\, j=1,2,\ldots,k} |x_1^{n_1} x_2^{n_2}\cdots x_k^{n_k} - P(x_1, x_2, \ldots, x_k)| = 2^{-N+k} \tag{23}$$

where $N=\sum_{j=1}^k n_j$.

Remark. The above result has been extended to the case where each x_j ranges over the union of $[-b_j, -a_j]$ and $[a_j, b_j]$, $j=1, 2, \ldots, k$.

Reddy [21]. Let $0\leqslant a<b$. Then there exists a real rational function of the form $P(x)/Q(x)$, where $P(x)$ is a polynomial of degree at most $2n-2$ and $Q(x)$ is of degree $2s$, for which we have in the union of $[-b, -a]$, and $[a, b]$,

$$\left\| x^{2n} - \frac{P(x)}{Q(x)} \right\|_{L_\infty} \leqslant (b^2-a^2)\, 2^{-2n+1} (A_2 - 2A_1 + A_0)^{-1} \tag{24}$$

where

$$A_j = \sum_{l=0}^{[(m-n-j)/2]} (-1)^l \binom{m-j-l-1}{n-1} \binom{m-j-l-n}{l} \left(\frac{2b^2+2a}{b^2-a^2} \right)^{m-j-n-2l},$$

$$m = 2n+2s.$$

Let $P(X)$ and $q(x)$ be any polynomials of degrees at most $(2n-1)$ and $2s$ respectively. Then we have in the union of the intervals $[-b, -a]$, $[a, b]$,

$$\left\| x^{2n} - \frac{P(x)}{q(x)} \right\|_{L_\infty} > \frac{(b^2-a^2)^n\, s!(2n)!(s+n)^{-1}}{[T_s((b^2+a^2)/(b^2-a^2))]\, 2^{2n}(s+2n-1)!}. \tag{25}$$

Reddy [21]. There is a monotonic polynomial $P^*(x)$ of degree at most $(2n-2k-1)$ $(0\leqslant k\leqslant n-1)$ for which we have in the union of $[-b, -a]$ and $[a, b]$,

$$\|x^{2n+1} - P^*(x)\|_{L_\infty} \leqslant (3b-2a)(2n+1) \sum_{m=n-k}^{n} |A_m| \tag{26}$$

where

$$A_{n-h} = \frac{(b^2-a^2)^n}{2^{2n-1}} \sum_{i=0}^{[h/2]} \binom{n-h+2i}{i} \binom{n}{n-h+2i} \left(\frac{2b^2+2a^2}{b^2-a^2} \right)^{h-2i}$$

Now we turn to the case of $|x|$.

Bernstein [2]. There exists a polynomial $P^*(x)$ of degree at most $2n$, for which

$$\| |x| - P^*(x) \|_{L_\infty[-1,1]} \leqslant \frac{1}{2n+1}. \tag{27}$$

He also established that for every real polynomial $P(x)$ of degree at most $2n$,

$$\| |x| - P(x) \|_{L_\infty[-1,1]} \geqslant \frac{1}{4(2n-1)(\sqrt{2}+1)}. \tag{28}$$

Newman ([10, p. 11]). For every $n \geqslant 4$, there is a real rational function $r(x)$ of degree $n+1$, for which

$$\| |x| - r(x) \|_{L_\infty[-1,1]} \leqslant 3e^{-\sqrt{n}}. \tag{29}$$

On the other hand, he has shown that for every real rational function $r(x)$ of degree at most n,

$$\| |x| - r(x) \|_{L_\infty[-1,1]} \geqslant \tfrac{1}{2} e^{-9\sqrt{n}}. \tag{30}$$

Gončar has derived Newman's result in a much sharper form from an earlier result of Zolotarev. It is interesting to note that, Zolotarev's work was published in 1877, also prior to Weierstrauss's approximation theorem.

Gončar ([8, p. 447]).

$$e^{-(\pi\sqrt{2}+\varepsilon)} < E_{n,n}(|x|) < e^{-((\pi/2)-\varepsilon)\sqrt{n}}, \qquad n \geqslant n_0(\varepsilon). \tag{31}$$

$$\left(E_{n,n}(f) = \inf_{a_j, b_j \text{ real}} \left\| f(x) - \left[\sum_{j=0}^{n} a_j x^j \Big/ \sum_{j=0}^{n} b_j x^j \right] \right\|_{L_\infty[-1,1]} \right)$$

Bulanov ([5], p.276).

$$\lim_{n \to \infty} (E_{n,n}(|x|))^{1/\sqrt{n}} = e^{-\pi}.$$

More precisely, for $n = 0, 1, 2, \ldots,$ and any $M > 0$

$$E_{n,n}(|x|) \geqslant e^{-\pi(n+1)^{1/2}}, \tag{32}$$

$$E_{n,n}(|x|) \leqslant e^{-\pi\sqrt{n}(1-\Delta(n))} \tag{33}$$

where $\Delta(n) = 0(n)$.

Vjačeslavov ([24, p. 680]).

$$E_{n,n}(|x|) \leqslant Ane^{-\pi\sqrt{n}}. \tag{34}$$

Where A is an absolute constant.

Boehm ([3, p. 396]).
There is a real polynomial $P^*(x)$ of degree n for which

$$\left\| |x| - \frac{1}{P^*(x)} \right\|_{L_\infty[-1,1]} \leqslant \frac{1+\pi}{n^{1/3}}. \tag{35}$$

Lungu ([9, p. 810]).
There is a real polynomial $P^*(x)$ of degree n for which

$$\left\| |x| - \frac{1}{P^*(x)} \right\|_{L_\infty[-1,1]} \leqslant (\log n/n). \tag{36}$$

Lungu ([9, p. 810]).
For every real polynomial $P(x)$ of degree at most n,

$$\left\| |x| - \frac{1}{P(x)} \right\|_{L_\infty[-1,1]} \geqslant \frac{1}{16n}. \tag{37}$$

Newman and Reddy ([14, p. 232]).
There exists a polynomial $P^*(x)$ of degree at most n, for which

$$\left\| |x| - \frac{1}{P^*(x)} \right\|_{L_\infty[-1,1]} \leqslant \frac{\pi^2}{2n}. \tag{38}$$

Erdos, Newman, and Reddy ([6, p. 137]).
There exists a polynomial $P^*(x)$ of degree n, for which

$$\left\| x - \frac{1}{P^*(x)} \right\|_{L_\infty[0,1]} \leqslant \frac{4}{n^2}. \tag{39}$$

Reddy [21]. Let k be any integer $\geqslant 1$. Then for each real polynomial $P(x_1, x_2, x_3, \ldots, x_k)$ of degree $2n_1, 2n_2, \ldots, 2n_k$, respectively, in the variables $x_1, x_2, \ldots, x_k$, we have for some constant c, depending only on k,

$$\left\| |x_1 x_2 x_3 \cdots x_k| - \frac{1}{P(x_1, x_2, \ldots, x_k)} \right\| \geqslant \frac{c_1}{n_1 n_2 \cdots n_k}. \tag{40}$$

Here $\|\cdot\|$ is the uniform norm over the cube $[-1, 1] \times [-1, 1] \times \cdots [-1, 1]$.

Reddy [19].
Let $S_n(x)=\sum_{i=0}^{n}\binom{2i}{i}\left(\frac{x}{4}\right)^i$. Then

$$\left\|\sqrt{1-x}-\frac{S_n(1)}{(1+S_n(1))\,S_n(x)}\right\|_{L_\infty[0,1]}\leqslant\frac{1}{1+S_n(1)}.$$

Let $R(x)=P(x)/Q(x)$, where $P(x)=\sum_{i=0}^{n}a_ix^i$, $a_0>0$, $a_i\geqslant 0$ for $1\leqslant i\leqslant n$, and $Q(x)=\sum_{i=0}^{n}b_ix^i$, $b_0>0$ and $0\leqslant b_i\leqslant b_0\binom{2i}{i}4^{-i}$ for $1\leqslant i\leqslant n$. Then

$$\|\sqrt{1-x}-R(x)\|_{L_\infty[0,1]}\geqslant\frac{1}{1+S_n(1)}.$$

Recently Ferguson and Szabados [25] have shown that, if $|x|$ is uniformly approximated on $[-1,1]$ by rational functions with integral coefficients with an error ε, $0<\varepsilon<\frac{1}{4}$, then at least one of the coefficients in greater than $(25\varepsilon)^{-1/2}$ in absolute value. By adopting a slightly different approach, Reddy has replaced $(25\varepsilon)^{-\frac{1}{2}}$ by $(7\varepsilon)^{-1}$.

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